

Variational Inequality Approach to Stochastic Nash Equilibrium Problems with an Application to Cournot Oligopoly

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Abstract

In this note we investigate stochastic Nash equilibrium problems by means of monotone variational inequalities in probabilistic Lebesgue spaces. We apply our approach to a class of oligopolistic market equilibrium problems where the data are known through their probability distributions.

Keywords: stochastic Nash equilibrium; Cournot oligopoly; stochastic variational inequalities; monotone operator

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1 Introduction

In this paper we deal with stochastic Nash equilibrium problems (SNEPs) which we analyze using the powerful tool of stochastic variational inequalities (SVIs). As an important application of the considered SNEP model, we investigate the oligopolistic market equilibria with uncertain data. In the present contribution, our objective is to establish a connection between general SNEPs and SVIs and propose a model of oligopolistic markets where the cost functions are not necessarily quadratic and the demand price is not restricted to be linear.

We emphasize that in the deterministic framework, it is well known that oligopolistic market equilibria are particular cases of Nash equilibria and that Nash equilibrium

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problems are equivalent to variational inequality problems under suitable differentiability hypotheses (see [1] for the infinite-dimensional case, and [14] for a finite-dimensional setting). Recently, some authors used a Lebesgue space formulation of oligopoly models to introduce time-dependent data ([2–4]). We consider SNEPs where the data are affected by a certain degree of uncertainty here, for example the case that the data are known by their probabilistic measures only. To provide a theoretical justification, we provide a formulation of SNEPs in Lebesgue spaces with probability measure, and then derive the associated SVIs. This mechanism allows us to exploit the recently developed tools of theory of stochastic variational inequalities in Lebesgue spaces (see e.g. [16–19]). We remark that in recent years many researchers devoted their efforts to SVIs and SNEPs [6, 7, 11, 15, 22, 27, 28, 30–32, 34]. However, these approaches differ from ours. They rely on defining a deterministic representative of the original stochastic variational inequality, and then use sample-average approximation techniques to get an estimate of the solution. In our previous work [21] we have done a comparative study to analyze different approaches for a traffic equilibrium problem, and the work illustrated different solution concepts and numerical methods.

This work is organized in 7 sections. In Section 2 we formulate the Nash equilibrium problem in a Lebesgue space with probability measure, and derive its equivalent stochastic variational inequality under suitable hypotheses. In Sections 3 and 4 we recall some theoretical results from [17, 19] together with a description of the approximation procedure used for the solution of the stochastic variational inequality. In Section 5 we propose a model of Cournot oligopoly with uncertain data and discuss the hypotheses needed to exploit the theory of stochastic variational inequalities. Section 6 is devoted to numerical examples: we introduce a stochastic version of a class of utility functions widely used in the literature which yield to nonlinear monotone stochastic variational inequalities. The final section contains summary of results and an outline of future research directions.

2 Stochastic Nash games and variational inequalities

Let (Ω, \mathcal{A}, P) be a probability space and consider a noncooperative game with m players each acting in a selfish manner in order to maximize their individual welfare. For P -almost every ω , each player i has a strategy vector $q_i = (q_{i1}, \dots, q_{im}) \in X_i(\omega)$, where $X_i(\omega) \subset \mathbb{R}^n$ is a convex and closed set, and a utility (or welfare) function

$$w_i : \Omega \times X_1(\omega) \times X_2(\omega) \times \dots \times X_m(\omega) \rightarrow \mathbb{R}.$$

He/she chooses his/her strategy vector q_i so as to maximize w_i , given the moves $(q_j)_{j \neq i}$ of the other players. We will use the notation

$$q_{-i} = (q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_m), \quad q = (q_i, q_{-i}).$$

Definition 2.1 *A stochastic Nash equilibrium is a random vector $q^*(\omega) = (q_1^*(\omega), \dots, q_m^*(\omega)) \in X(\omega) = X_1(\omega) \times X_2(\omega) \times \dots \times X_m(\omega)$, such that P -a.s. (almost surely):*

$$w_i(\omega, q_i^*(\omega), q_{-i}^*(\omega)) \geq w_i(\omega, q_i, q_{-i}^*(\omega)), \quad \forall q_i \in X_i(\omega), \forall i \in \{1, \dots, m\}. \quad (1)$$

The following theorem relates Nash equilibrium problems and variational inequalities. For its proof it suffices to apply the classical finite-dimensional proof, for each fixed value of the random parameter ω .

Theorem 2.1 *Let $w_i(\omega, \cdot) \in C^1(\mathbb{R}^{mn}), \forall i$, and concave with respect to q_i . Let $F : \Omega \times \mathbb{R}^{mn} \rightarrow \mathbb{R}^{mn}$ be the mapping built with the partial gradients of the utility functions as follows:*

$$F(\omega, q) = (-\nabla_{q_1} w_1(\omega, q), \dots, -\nabla_{q_m} w_m(\omega, q)).$$

Then, $q^(\omega) \in X(\omega)$ is a stochastic Nash equilibrium if and only if, $P - a.s.$, it satisfies the variational inequality:*

$$F(\omega, q^*(\omega)) \cdot (q - q^*(\omega)) = \sum_{r=1}^m -\nabla_{q_r} w_r(\omega, q^*(\omega)) \cdot (q_r - q_r^*(\omega)) \geq 0, \forall q \in X(\omega). \quad (2)$$

Problems (1) and (2) are parametric versions of the deterministic problems, where the random parameter ω belongs to the given sample space Ω . A solution $q^*(\omega)$ of these problems is a random vector. From a statistical point of view it is important that $q^*(\omega)$ has finite first and second moments. As a consequence, we introduce integral versions of (1) and (2).

Thus, let $p \geq 2, \forall i$ define the set:

$$K_i = \{v \in L^p(\Omega, P, \mathbb{R}^n) : v(\omega) \in X_i(\omega), P - a.s.\}$$

and consider the problem of finding $u^* \in L^p(\Omega, P, \mathbb{R}^{mn})$ such that $\forall i \in \{1, \dots, m\}$ one has:

$$\int_{\Omega} w_i(\omega, u^*(\omega)) dP_{\omega} = \max_{u_i \in K_i} \int_{\Omega} w_i(\omega, u_i(\omega), u_{-i}^*(\omega)) dP_{\omega}. \quad (3)$$

The associated variational inequality problem is the following:

Find $u^* \in K_P$:

$$\int_{\Omega} \sum_{r=1}^m -\nabla_{q_r} w_r(\omega, u^*(\omega)) \cdot (u_r(\omega) - u_r^*(\omega)) dP_{\omega} \geq 0 \quad \forall u \in K_P \quad (4)$$

where

$$K_P = K_1 \times \dots \times K_m.$$

In (3) we have introduced, $\forall i$, the functional $J_i : L^p(\Omega, P, \mathbb{R}^{mn}) \rightarrow \mathbb{R}$ through:

$$J_i(u_1, \dots, u_m) = \int_{\Omega} w_i((\omega), u_1(\omega), \dots, u_m(\omega)) dP_{\omega} \quad (5)$$

In order that this functional be well defined and to work with (3) and (4), we shall impose a set of assumptions on the functions w_i , namely:

- (a) For all $i \in \{1, \dots, m\}$, $w_i(\cdot, q)$ be a random variable with respect to the sigma-algebra defined on Ω , $\forall q$, and $w_i(\omega, \cdot) \in C^1(\mathbb{R}^{mn})$ P -a.s.
- (b) For all $i \in \{1, \dots, m\}$, $w_i(\omega, 0) \in L^1(\Omega, P)$.

- (c) For all $i \in \{1, \dots, m\}$, $w_i(\omega, q)$ be concave with respect to q_i , $P - a.s.$, for all fixed values of q_{-i} .
- (d) For all $i \in \{1, \dots, m\}$, $|\nabla_q w_i(\omega, q)| \leq \alpha(\omega) + \beta_i(\omega)|q|^{p-1}$, where $\beta_i \in L^\infty(\Omega, P)$ and $\alpha \in L^{p'}(\Omega, P)$, $p' = p/p - 1$.

We are now in a position to prove a simple lemma which is fundamental for the sequel.

Theorem 2.2 *Let assumptions a)-d) be fulfilled. Then, for all $i \in \{1, \dots, m\}$ the functional $J_i(u) = J_i(u_i, u_{-i})$ is well defined on $L^p(\Omega, P, \mathbb{R}^{mn})$ and concave with respect to the variable $u_i \in L^p(\Omega, P, \mathbb{R}^n)$ for each fixed u_{-i} . Moreover J_i is Gateaux-differentiable with respect to u_i , for each u_{-i} and its derivative is given by:*

$$D_i J_i(u_i, u_{-i})(v_i) = \int_{\Omega} \nabla_{q_i} w_i(\omega, u_i, u_{-i}) \cdot v_i(\omega) dP_{\omega} = \quad (6)$$

$$\int_{\Omega} \sum_{r=1}^n \left(\frac{\partial}{\partial q_r} (w_i(\omega, u(\omega))) \right) v_{ir}(\omega) dP_{\omega}, \quad \forall v_i = (v_{i1}, \dots, v_{in}) \in L^p(\Omega, P, \mathbb{R}^n)$$

Proof. First, we show that the functional J_i is well defined for all i . Thus, for P -almost every $\omega \in \Omega$ apply Lagrange Theorem to the function $w_i(\omega, q)$, with respect to the interval of endpoints $0, q$. We get that $\exists \xi \in \mathbb{R}^{mn}$, $|\xi| < |q|$ such that:

$$|w_i(\omega, q)| \leq |w_i(\omega, 0)| + |\nabla_q w_i(\omega, \xi)| |q| \leq |w_i(\omega, 0)| + |\alpha(\omega)| |q| + \beta_i(\omega) |\xi|^{p-1} |q|.$$

Then, $\forall u \in L^p(\Omega, P, \mathbb{R}^{mn})$ we get

$$|w_i(\omega, u(\omega))| \leq |w_i(\omega, 0)| + |\alpha(\omega)| |u(\omega)|^{p-1} + \beta_i(\omega) |u(\omega)|$$

which shows that $w_i(\omega, u(\omega))$ belongs to $L^1(\Omega, P)$, hence J_i is well defined. The concavity of $J_i(u_i, u_{-i})$ with respect to u_i is a straightforward consequence of the analogous property of $w_i(\omega, q)$.

In order to prove that J_i is Gateaux-differentiable with respect to u_i , for each fixed u_{-i} , fix a point u_i , a direction v_i and for each $t \in]0, 1[$ consider the quotient:

$$\begin{aligned} & \frac{J_i(u_i + tv_i, u_{-i}) - J_i(u_i, u_{-i})}{t} \\ &= \int_{\Omega} \frac{1}{t} [w_i(\omega, u_i(\omega) + tv_i(\omega), u_{-i}(\omega)) - w_i(\omega, u_i(\omega), u_{-i}(\omega))] dP_{\omega} \\ &= \int_{\Omega} \nabla_{q_i} w_i(\omega, u_i(\omega) + th(\omega)v_i(\omega), u_{-i}(\omega)) \cdot v_i(\omega) dP_{\omega}, \end{aligned}$$

where $h : \Omega \rightarrow [0, 1]$ is a random variable. We obtain (6) because it is possible to pass to the limit under the integral sign for $t \rightarrow 0$. Indeed, since $w_i(\omega, \cdot)$ has continuous partial derivatives, it follows that for $t \rightarrow 0$, we get, $P - a.s.$:

$$\nabla_{q_i} w_i(\omega, u_i(\omega) + th(\omega)v_i(\omega), u_{-i}(\omega)) \cdot v_i(\omega) \longrightarrow \nabla_{q_i} w_i(\omega, u_i(\omega), u_{-i}(\omega)) \cdot v_i(\omega),$$

moreover

$$\begin{aligned} & |\nabla_{q_i} w_i(\omega, u_i(\omega) + th(\omega) v_i(\omega), u_{-i}(\omega)) \cdot v_i(\omega)| \\ & \leq |\alpha(\omega)| |v_i(\omega)| + \beta_i(\omega) (|u_i(\omega)| + |v_i(\omega)| + |u_{-i}(\omega)|)^{p-1}. \end{aligned}$$

At last, the fact that $D_i J_i(u)(\cdot)$ is a linear and continuous functional on $L^p(\Omega, P, \mathbb{R}^n)$ concludes the proof. \square

Once we have established the expression of the Gateaux derivative of J_i , consider, for each u , the operator $\Gamma(u) : L^p(\Omega, P, \mathbb{R}^{mn}) \rightarrow L^{p'}(\Omega, P, \mathbb{R}^{mn})$ defined by:

$$\Gamma(u) = (-D_1 J_1(u), \dots, -D_m J_m(u)).$$

Then, from the infinite dimensional theory of Nash equilibrium problems, we get (see e.g. [1]) that (3) is equivalent to

$$u^* \in K_P : \Gamma(u^*)(u - u^*) \geq 0, \forall u \in K_P,$$

which is nothing other than (4).

3 Stochastic variational inequalities in Lebesgue spaces

In the sequel we shall study SNEPs and, in particular, the oligopolistic market, through its equivalent variational inequality (4). As mentioned in the introduction, variational inequalities of this kind have been introduced quite recently and in this section we recall the main results useful for our application. A more comprehensive treatment can be found in [17–19]. In particular, we shall treat the case where the deterministic and random variable are separated and in this case an approximation procedure for the computation of the solution is presented.

Let (Ω, \mathcal{A}, P) be a probability space. Let $G, H : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be two given maps, let $b, c \in \mathbb{R}^k$ be fixed vectors, and let R and S be two real-valued random variables defined on Ω . Let λ be a random vector in \mathbb{R}^k , let D be random vector in \mathbb{R}^m , and let $A \in \mathbb{R}^{m \times k}$ be a given matrix. For $\omega \in \Omega$, we define a random set

$$M(\omega) := \{x \in \mathbb{R}^k : Ax \leq D(\omega)\}.$$

Consider the following stochastic variational inequality: For almost all $\omega \in \Omega$, find $\hat{x} := \hat{x}(\omega) \in M(\omega)$ such that

$$\langle S(\omega)G(\hat{x}) + H(\hat{x}), z - \hat{x} \rangle \geq \langle R(\omega)c + b, z - \hat{x} \rangle, \quad \text{for every } z \in M(\omega). \quad (7)$$

Variational inequality (7) holds pointwise on Ω , except a fixed null set depending on the solution \hat{x} . To facilitate the foregoing discussion, we set

$$F(\omega, x) := S(\omega)G(x) + H(x),$$

Let S, G and H be such that $F : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$ is a Carathéodory function. That is, for each fixed $x \in \mathbb{R}^k$, the function $F(\cdot, x)$ is measurable with respect to \mathcal{A} whereas for each

$\omega \in \Omega$ the function $F(\omega, \cdot)$ is continuous. We also assume that $F(\omega, \cdot)$ is monotone for every $\omega \in \Omega$:

$$\langle F(\omega, x) - F(\omega, y), x - y \rangle \geq 0, \quad \forall x, y, \forall \omega$$

If the equality sign holds only for $x = y$, then F is said strictly monotone and, in this case there is at most a solution to (7) which, under suitable conditions belongs to an L^p space for some $p \geq 2$.

A stronger form of monotonicity will be useful in the sequel:

Definition 3.1 F is strongly monotone, uniformly with respect to ω iff $\exists a > 0$:

$$\langle F(\omega, x) - F(\omega, y), x - y \rangle \geq a \|x - y\|^2, \quad \forall x, y, \forall \omega.$$

For this, we proceed to derive the integral formulation of (7). For a fixed $p \geq 2$, we define the reflexive Banach space $L^p(\Omega, P, \mathbb{R}^k)$ of random vectors V from Ω to \mathbb{R}^k such that the expectation (p -moment) is given by:

$$E^P \|V\|^p = \int_{\Omega} \|V(\omega)\|^p dP(\omega) < \infty.$$

For the subsequent development, we need the following growth condition

$$\|F(\omega, z)\| \leq \alpha(\omega) + \beta(\omega) \|z\|^{p-1}, \quad \forall z \in \mathbb{R}^k, \quad \text{for some } p \geq 2, \quad (8)$$

where $\alpha \in L^{p'}(\Omega, P)$ and $\beta \in L^\infty(\Omega, P)$.

Due to the above growth condition, the Nemitsky operator \hat{F} associated to F , acts from $L^p(\Omega, P, \mathbb{R}^k)$ to $L^{p'}(\Omega, P, \mathbb{R}^k)$, where $p^{-1} + p'^{-1} = 1$. Furthermore, we have

$$\hat{F}(V)(\omega) := F(\omega, V(\omega)), \quad \omega \in \Omega.$$

Assuming $D \in L_m^p(\Omega) := L^p(\Omega, P, \mathbb{R}^m)$, we introduce the following nonempty, closed and convex subset of $L_k^p(\Omega)$

$$M^P := \{V \in L_k^p(\Omega) : AV(\omega) \leq D(\omega), \quad P - a.s.\},$$

which is the L^p analogue of $M(\omega)$ defined above.

Let $S(\omega) \in L^\infty$, $0 < \underline{s} < S(\omega) < \bar{s}$, and $R(\omega) \in L^{p'}$. Equipped with these notation, we consider the following L^p formulation of (7). Find $\hat{U} \in M^P$ such that for every $V \in M^P$, we have

$$\int_{\Omega} \langle S(\omega) G(\hat{U}(\omega)) + H(\hat{U}(\omega)), V(\omega) - \hat{U}(\omega) \rangle dP(\omega) \geq \int_{\Omega} \langle b + R(\omega)c, V(\omega) - \hat{U}(\omega) \rangle dP(\omega). \quad (9)$$

To get rid of the abstract sample space Ω , we consider the joint distribution \mathbb{P} of the random vector (R, S, D) and work with the special probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$, where the dimension $d := 2 + m$. For simplicity, we assume that R, S and D are independent random vectors. We set

$$\begin{aligned} r &= R(\omega), \\ s &= S(\omega), \\ t &= D(\omega), \\ y &= (r, s, t). \end{aligned}$$

For each $y \in \mathbb{R}^d$, we define the set

$$M(y) := \{x \in \mathbb{R}^k : Ax \leq t\}.$$

The pointwise formulation of the variational inequality reads: Find \hat{x} such that $\hat{x}(y) \in M(y)$, \mathbb{P} - a.s., and the following inequality holds for \mathbb{P} - almost every $y \in \mathbb{R}^d$ and for every $x \in M(y)$, we have

$$\langle sG(\hat{x}(y)) + H(\hat{x}(y)), x - \hat{x}(y) \rangle \geq \langle rc + b, x - \hat{x}(y) \rangle. \quad (10)$$

In order to obtain the integral formulation of (10), consider the space $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ and introduce the closed and convex set

$$M_{\mathbb{P}} := \{v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : Av(r, s, t) \leq t, \mathbb{P} - a.s.\}.$$

With this terminology, we consider the variational inequality of finding $\hat{u} \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$ we have

$$\int_0^{\infty} \int_{\mathbb{R}^d} \langle sG(\hat{u}(y)) + H(\hat{u}(y)), v(y) - \hat{u}(y) \rangle d\mathbb{P}(y) \geq \int_0^{\infty} \int_{\mathbb{R}^d} \langle b + rc, v(y) - \hat{u}(y) \rangle d\mathbb{P}(y). \quad (11)$$

Remark 3.1 *Our approach and analysis extends readily to more general finite Karhunen-Loève expansions*

$$\lambda(\omega) = b + \sum_{l=1}^L R_l(\omega) c_l, \quad F(\omega, x) = H(x) + \sum_{l=1}^{L_F} S_l(\omega) G_l(x).$$

4 An Approximation Procedure by Discretization of Distributions

Without any loss of generality, we assume that $R \in L^q(\Omega, P)$ and $D \in L_m^p(\Omega, P)$ are non-negative (otherwise we can use the standard decomposition in the positive part and the negative part). Moreover, we assume that the support, the set of possible outcomes, of $S \in L^\infty(\Omega, P)$ is the interval $[\underline{s}, \bar{s}] \subset (0, \infty)$. Furthermore, we assume that the probability measures P_R , P_S , and P_D are continuous with respect to the Lebesgue measure, so that according to the theorem of Radon-Nikodym, they have the probability densities φ_R , φ_S , and φ_{D_i} , $i = 1, \dots, m$, respectively. Therefore, for $i = 1, \dots, m$, we have

$$\begin{aligned} \mathbb{P} &= P_R \otimes P_S \otimes P_D, \\ dP_R(r) &= \varphi_R(r) dr, \\ dP_S(s) &= \varphi_S(s) ds \\ dP_{D_i}(t_i) &= \varphi_{D_i}(t_i) dt_i. \end{aligned}$$

Notice that $v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ means that $(r, s, t) \mapsto \varphi_R(r) \varphi_S(s) \varphi_D(t) v(r, s, t)$ belongs to the Lebesgue space $L^p(\mathbb{R}^d, \mathbb{R}^k)$ with respect to the Lebesgue measure where

$$\varphi_D(t) := \prod_i \varphi_{D_i}(t_i).$$

Therefore, we can define the probabilistic integral variational inequality: Find $\hat{u} := \hat{u}(y) \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$, we have

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}_+^m} \langle s G(\hat{u}) + H(\hat{u}), v - \hat{u} \rangle \varphi_R(r) \varphi_S(s) \varphi_D(t) dy \geq \int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}_+^m} \langle b + r c, v - \hat{u} \rangle \varphi_R(r) \varphi_S(s) \varphi_D(t) dy.$$

For numerical approximation of the solution \hat{u} , we begin with a discretization of the space $X := L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. For this, we introduce a sequence $\{\pi_n\}_n$ of partitions of the support

$$\Upsilon := [0, \infty) \times [\underline{s}, \bar{s}) \times \mathbb{R}_+^m$$

of the probability measure \mathbb{P} induced by the random elements R, S , and D . For this, we set

$$\pi_n = (\pi_n^R, \pi_n^S, \pi_n^D),$$

where

$$\begin{aligned} \pi_n^R &:= (r_n^0, \dots, r_n^{N_n^R}), \\ \pi_n^S &:= (s_n^0, \dots, s_n^{N_n^S}), \\ \pi_n^{D_i} &:= (t_{n,i}^0, \dots, t_{n,i}^{N_{n,i}^{D_i}}) \\ 0 &= r_n^0 < r_n^1 < \dots < r_n^{N_n^R} = n \\ \underline{s} &= s_n^0 < s_n^1 < \dots < s_n^{N_n^S} = \bar{s} \\ 0 &= t_{n,i}^0 < t_{n,i}^1 < \dots < t_{n,i}^{N_{n,i}^{D_i}} = n \quad (i = 1, \dots, m) \\ |\pi_n^R| &:= \max\{r_n^j - r_n^{j-1} : j = 1, \dots, N_n^R\} \rightarrow 0 \quad (n \rightarrow \infty) \\ |\pi_n^S| &:= \max\{s_n^k - s_n^{k-1} : k = 1, \dots, N_n^S\} \rightarrow 0 \quad (n \rightarrow \infty) \\ |\pi_n^{D_i}| &:= \max\{t_{n,i}^{h_i} - t_{n,i}^{h_i-1} : h_i = 1, \dots, N_{n,i}^{D_i}\} \rightarrow 0 \quad (i = 1, \dots, m; n \rightarrow \infty). \end{aligned}$$

These partitions give rise to the exhausting sequence $\{\Upsilon_n\}$ of subsets of Υ , where each Υ_n is given by the finite disjoint union of the intervals:

$$I_{jkh}^n := [r_n^{j-1}, r_n^j) \times [s_n^{k-1}, s_n^k) \times I_h^n,$$

where we use the multi-index $h = (h_1, \dots, h_m)$ and

$$I_h^n := \prod_{i=1}^m [t_{n,i}^{h_i-1}, t_{n,i}^{h_i}).$$

For each $n \in \mathbb{N}$, we consider the space of the \mathbb{R}^l -valued step functions ($l \in \mathbb{N}$) on Υ_n , extended by 0 outside of Υ_n :

$$X_n^l := \{v_n : v_n(r, s, t) = \sum_j \sum_k \sum_h v_{jkh}^n 1_{I_{jkh}^n}(r, s, t), v_{jkh}^n \in \mathbb{R}^l\}$$

where 1_I denotes the $\{0, 1\}$ -valued characteristic function of a subset I .

To approximate an arbitrary function $w \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R})$, we employ the mean value truncation operator μ_0^n associated to the partition π_n given by

$$\mu_0^n w := \sum_{j=1}^{N_n^R} \sum_{k=1}^{N_n^S} \sum_h (\mu_{jkh}^n w) 1_{I_{jkh}^n}, \quad (12)$$

where

$$\mu_{jkh}^n w := \begin{cases} \frac{1}{\mathbb{P}(I_{jkh}^n)} \int_{I_{jkh}^n} w(y) d\mathbb{P}(y) & \text{if } \mathbb{P}(I_{jkh}^n) > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, for a L^p vector function $v = (v_1, \dots, v_l)$, we define

$$\mu_0^n v := (\mu_0^n v_1, \dots, \mu_0^n v_l).$$

The basic property of the mean value truncation operator is expressed in the following lemma (see [17]).

Lemma 4.1 *For any fixed $l \in \mathbb{N}$, the linear operator $\mu_0^n : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l) \rightarrow L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$ is bounded with $\|\mu_0^n\| = 1$ and for $n \rightarrow \infty$, μ_0^n converges pointwise in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$ to the identity.*

To construct approximations for

$$M_{\mathbb{P}} = \{v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : Av(r, s, t) \leq t, \mathbb{P} - a.s.\},$$

we introduce the orthogonal projector $q : (r, s, t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$ and define for each elementary cell I_{jkh}^n ,

$$\bar{q}_{jkh}^n = (\mu_{jkh}^n q) \in \mathbb{R}^m, \quad (\mu_0^n q) = \sum_{jkh} \bar{q}_{jkh}^n 1_{I_{jkh}^n} \in X_n^m.$$

This leads to the following sequence of convex and closed sets of the polyhedral type:

$$M_{\mathbb{P}}^n := \{v \in X_n^k : Av_{jkh}^n \leq \bar{q}_{jkh}^n, \forall j, k, h\}.$$

It has been proven (see [17]) that the sequence $\{M_{\mathbb{P}}^n\}$ approximate the set $M_{\mathbb{P}}$ in the sense of Mosco ([25]). In order to approximate the random variables R and S , we introduce

$$\begin{aligned} \rho_n &= \sum_{j=1}^{N_n^R} r_n^{j-1} 1_{[r_n^{j-1}, r_n^j)} \in X_n \\ \sigma_n &= \sum_{k=1}^{N_n^S} s_n^{k-1} 1_{[s_n^{k-1}, s_n^k)} \in X_n, \end{aligned}$$

where

$$\begin{aligned}\sigma_n(r, s, t) &\rightarrow \sigma(r, s, t) = s, \quad \text{in } L^\infty(\mathbb{R}^d, \mathbb{P}) \\ \rho_n(r, s, t) &\rightarrow \rho(r, s, t) = r, \quad \text{in } L^p(\mathbb{R}^d, \mathbb{P}).\end{aligned}$$

Combining the above ingredients, for $n \in \mathbb{N}$, we consider the following discretized variational inequality: Find $\hat{u}_n := \hat{u}_n(y) \in M_{\mathbb{P}}^n$ such that for every $v_n \in M_{\mathbb{P}}^n$, we have

$$\int_0^\infty \int_{\mathbb{R}^d} \langle \sigma_n(y) G(\hat{u}_n) + H(\hat{u}_n), v_n - \hat{u}_n \rangle d\mathbb{P}(y) \geq \int_0^\infty \int_{\mathbb{R}^d} \langle b + \rho_n(y) c, v_n - \hat{u}_n \rangle d\mathbb{P}(y). \quad (13)$$

It turns out that (13) can be split in a finite number of finite dimensional variational inequalities: For every $n \in \mathbb{N}$, and for every j, k, h find $\hat{u}_{jkh}^n \in M_{jkh}^n$ such that

$$\langle \tilde{F}_k^n(\hat{u}_{jkh}^n), v_{jkh}^n - \hat{u}_{jkh}^n \rangle \geq \langle \tilde{c}_j^n, v_{jkh}^n - \hat{u}_{jkh}^n \rangle, \quad \text{for every } v_{jkh}^n \in M_{jkh}^n, \quad (14)$$

where

$$\begin{aligned}M_{jkh}^n &:= \{v_{jkh}^n \in \mathbb{R}^k : Av_{jkh}^n \leq \bar{q}_{jkh}^n\}, \\ \tilde{F}_k^n &:= s_n^{k-1} G + H \\ \tilde{c}_j^n &:= b + r_n^{j-1} c.\end{aligned}$$

Clearly, we have

$$\hat{u}_n = \sum_j \sum_k \sum_h \hat{u}_{jkh}^n 1_{I_{jkh}^n} \in X_n^k.$$

Now, we can state the following convergence result (whose proof can be found in [19]).

Theorem 4.1 *Assume that $F(\omega, \cdot)$ is strongly monotone uniformly with respect to $\omega \in \Omega$. Then the sequence \hat{u}_n generated by the substitute problems in (13) converges strongly in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ for $n \rightarrow \infty$ to the unique solution \hat{u} of (11).*

Remark 4.1 *Looking carefully at the proof in [19], we can deduce that if the uniform strong monotonicity hypothesis is not satisfied, and F is only monotone, but we know that the solution is unique we obtain weak convergence of \hat{u}_n to \hat{u} . This implies convergence of the approximate mean values to the exact mean value of the solution.*

5 The stochastic oligopoly model

In this section, we propose a model of oligopolistic market with uncertain data and show that the theoretical and numerical tools developed in the previous sections can be successfully applied to the model. The classical oligopolistic market equilibrium problem is a Nash game with a special structure and it was first introduced by A. Cournot [10] a long time ago. Recent years have witnessed a renewed interest in

oligopoly theory, and many specific cases of oligopolistic markets have been studied in detail, for instance the electricity market (see e.g. [8, 9], and [5] for a model based on real industrial data).

We consider here the case in which m players are the producers of the same commodity. The quantity produced by firm i is denoted by q_i so that $q \in \mathbb{R}^m$ denotes the global production vector. Let (Ω, \mathcal{A}, P) be a probability space and for every $i \in \{1, \dots, m\}$ consider functions $f_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and $p : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$. More precisely, $f_i(\omega, q_i)$ represents the cost of producing the commodity for firm i , and is assumed to be, P -a.s., nonnegative, increasing and C^1 , while $p(\omega, q_1 + \dots + q_m)$ represents the demand price associated with the commodity. For P -almost every $\omega \in \Omega$, p is assumed nonnegative, increasing and C^1 . The resulting welfare function w_i is assumed to be concave with respect to q_i . We also assume that all these functions are random variables w.r.t. ω , i.e. they are measurable with respect to the probability measure P on Ω . In this way, we cover the possibility that both the production cost and the demand price are affected by a certain degree of uncertainty, or randomness. Thus, the welfare (or utility) function of player i , representing the net revenue, is given by:

$$w_i(\omega, q_1, \dots, q_m) = p(\omega, q_1 + \dots + q_m)q_i - f_i(\omega, q_i). \quad (15)$$

Although many models assume no bounds on the production, in a more realistic model the production capability is bounded from above and we also allow these upper bounds to be random variables: $0 \leq q_i \leq \bar{q}_i(\omega)$. Thus, the specific Nash equilibrium problem associated with this model takes the following form:

For P -a.e. $\omega \in \Omega$, find $q^*(\omega) = (q_1^*(\omega), \dots, q_m^*(\omega))$:

$$w_i(\omega, q^*(\omega)) = \max_{0 \leq q_i \leq \bar{q}_i(\omega)} \{p(\omega, q_i + \sum_{j \neq i} q_j^*(\omega))q_i, -f_i(\omega, q_i)\}, \forall i \in \{1, \dots, m\}. \quad (16)$$

In order to write the equivalent variational inequality, consider, $\forall \omega$, a closed and convex subset of \mathbb{R}^m :

$$K(\omega) = \{(q_1, \dots, q_m) : 0 \leq q_i \leq \bar{q}_i(\omega), \forall i\}$$

and define the functions

$$\begin{aligned} F_i(\omega, q) &:= \frac{\partial f_i(\omega, q_i)}{\partial q_i} - \frac{\partial p(\omega, \sum_{j=1}^m q_j)}{\partial q_i} q_i - p(\omega, \sum_{j=1}^m q_j) \\ &= f'_i(\omega, q_i) - p'(\omega, Q)q_i - p(\omega, Q), \quad (Q = \sum_{j=1}^m q_j). \end{aligned} \quad (17)$$

The Nash problem is then equivalent to the following variational inequality: for P -a.e. $\omega \in \Omega$, find $q^*(\omega) \in K(\omega)$ such that

$$\sum_{i=1}^m \left[\frac{\partial f_i(\omega, q_i^*(\omega))}{\partial q_i} - \frac{\partial p(\omega, \sum_{j=1}^m q_j^*(\omega))}{\partial q_i} q_i - p(\omega, \sum_{j=1}^m q_j^*(\omega)) \right] (q_i - q_i^*(\omega)) \geq 0 \quad (18)$$

$\forall q \in K(\omega)$.

Remark 5.1 Since $F(\omega, \cdot)$ is continuous, and $K(\omega)$ is convex and compact, problem (18) is solvable for almost every $\omega \in \Omega$, due to the Stampacchia's theorem. In the case that the production capability is assumed unbounded some additional hypotheses (i.e. coercivity, see e.g. [24]) have to be present to ensure the solvability of (18).

Moreover, we assume that $F(\omega, \cdot)$ is monotone, i.e.:

$$\sum_{i=1}^m (F_i(\omega, q) - F_i(\omega, q'))(q_i - q'_i) \geq 0 \quad \forall \omega \in \Omega, \quad \forall q, q' \in \mathbb{R}^m.$$

(recall that F is said to be strictly monotone if the equality holds only for $q = q'$ and in this case (18) has a unique solution). It is noteworthy that some classes of utility functions widely used in the economic literature enjoy some form of monotonicity (see section 6).

Now we are interested in computing statistical quantities associated with the solution $q^*(\omega)$, in particular its mean value. For this purpose, in accordance with the general scheme of Section 2, we consider a Lebesgue space formulation of problems (18): Find $u^* \in K$ such that

$$\int_{\Omega} \sum_{i=1}^m \left[\frac{\partial f_i(\omega, u_i^*(\omega))}{\partial q_i} - \frac{\partial p(\omega, \sum_{j=1}^m u_j^*(\omega))}{\partial q_i} u_i - p(\omega, \sum_{j=1}^m u_j^*(\omega)) \right] \times (u_i(\omega) - u_i^*(\omega)) dP_{\omega} \geq 0, \quad (19)$$

where

$$K = \{u \in L^P(\Omega, P, \mathbb{R}^m) : 0 \leq u_i(\omega) \leq \bar{q}_i(\omega)\}, \quad \bar{q}_i \in L^P(\omega, P).$$

Since the stochastic oligopolistic market problem will be studied through (19), we ensure its solvability by the following theorem:

Theorem 5.1 Let $f_i(\cdot, q_i), p(\cdot, \sum_{j=1}^m q_j)$ be measurable, and $f_i(\omega, \cdot), d_i(\omega, \cdot)$ be of class C^1 . Let F be strictly monotone and satisfy the growth condition $d)$ of Section 2. Then (19) admits a unique solution.

Proof. Under our assumptions, $F : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a Carathéodory function and it is well known that for each measurable function $u(\omega)$, the function $F(\omega, u(\omega))$ is also measurable. Under the growth condition $d)$ the superposition operator $\hat{F} : u(\omega) \rightarrow F(\omega, u(\omega))$ maps $L^P(\Omega, P, \mathbb{R}^m)$ in $L^{P'}(\Omega, P, \mathbb{R}^m)$ and is continuous, being P a probability measure. Moreover, the uniform strong monotonicity of F implies the strong monotonicity of \hat{F} . The set K_P is convex, closed and (norm) bounded, hence weakly compact. Then, monotone operator theory applies and (19) admits a unique solution (see e.g. [24] for a recent survey on existence theorems which also includes the case of unbounded sets). \square

Now, in view of the numerical approximation of the solution, we further specialize our model and assume that the random and the deterministic part of the operator can be separated. Thus, we assume that the price can be affected by two random perturbations $\alpha(\omega)$ and $S(\omega)$ such that:

$$p(\omega, Q) = S(\omega)p(Q) + \alpha(\omega),$$

while the cost functions are of the type:

$$f_i(\omega, q_i) = \beta_i(\omega)f_i(q_i) + g_i(q_i),$$

that is, the cost functions consists of a deterministic term g_i and a term, (still denoted by f_i with an abuse of notation), which is *modulated* by the random perturbation β_i . Here α, β_i are real random variables, with $0 < \underline{s} \leq S(\omega) \leq \bar{s}$, $0 < \underline{\beta}_i \leq \beta_i(\omega) \leq \bar{\beta}_i$.

As a consequence, the operator F takes the form:

$$F_i(\omega, q) = \beta_i(\omega) \frac{\partial f_i(q_i)}{\partial q_i} + \frac{\partial g_i(q_i)}{\partial q_i} - S(\omega) p\left(\sum_{j=1}^m q_j\right) - \alpha(\omega) - S(\omega) \frac{\partial p(\sum_{j=1}^m q_j)}{\partial q_i} q_i.$$

Furthermore, we assume that F is uniformly strongly monotone, $w_i(\omega, 0) \in L^1(\Omega)$, and the growth condition d) of Section 2 is satisfied.

Now, according to the methodology explained in Section 3, we will work with the probability distributions induced on the images of the functions: $A = \alpha(\omega)$, $s = S(\omega)$, $B_i = \beta_i(\omega)$, $Q_i = \bar{q}_i(\omega)$. Thus, let $y = (A, s, B, Q)$ and consider the probability space $(\mathbb{R}^d, \mathcal{B}, \mathbb{P})$ with $d = 2 + 2m$, where \mathcal{B} is the Borel sigma-algebra on \mathbb{R}^d . In order to formulate our problem in the image space, we introduce the closed convex set $K_{\mathbb{P}}$ by:

$$K_{\mathbb{P}} = \{u \in L^2(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^m) : 0 \leq u_i(A, s, B, Q) \leq \bar{Q}_i, \forall i, \mathbb{P} - a.s.\}.$$

We assume that all the random variables are independent and that each probability distribution is characterized by its density φ . Thus, we have $\mathbb{P} = \mathbb{P}_A \otimes \mathbb{P}_s \otimes \mathbb{P}_B \otimes \mathbb{P}_Q$, $dP_{\alpha}(A) = \varphi_{\alpha}(A)dA$, $dP_s(s) = \varphi_s(s)ds$, $dP_{\beta}(B) = \varphi_{\beta}(B)dB$, $dP_{\bar{q}}(Q) = \varphi_{\bar{q}}(Q)dQ$, where we used the compact notation $\varphi_x(X) = \prod_{i=1}^n \varphi_{x_i}(X_i)$. Thus, we obtain the following problem: Find $u^* \in K_{\mathbb{P}}$ such that $\forall u \in K_{\mathbb{P}}$

$$\begin{aligned} & \int_{\underline{s}}^{\bar{s}} \int_{\underline{\beta}}^{\bar{\beta}} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \sum_{i=1}^m \left[B_i \frac{\partial f_i(u_i^*(A, s, B, Q))}{\partial q_i} + \frac{\partial g_i(u_i^*(A, s, B, Q))}{\partial q_i} - p\left(\sum_{j=1}^m u_j^*(A, s, B, Q)\right) \right. \\ & \left. - A - \frac{\partial p(\sum_{j=1}^m u_j^*(A, s, B, Q))}{\partial q_i} u_i^*(A, s, B, Q) \right] \times \\ & (u_i(A, B, Q) - u_i^*(A, s, B, Q)) \varphi_{\alpha}(A) \varphi_s(s) \varphi_{\beta}(B) \varphi_{\bar{q}}(Q) ds dA dB dQ \geq 0, \end{aligned} \quad (20)$$

where the symbol $\int_{\underline{\beta}}^{\bar{\beta}}$ represents the m integrals $\int_{\underline{\beta}_i}^{\bar{\beta}_i}$. This formulation is suitable for the approximation procedure based on discretization and truncation explained in Section 4.

6 A class of utility functions and numerical examples

In this section we consider a modified and random version of a class of utility functions introduced by Murphy, Sheraly and Soyster in [23] and successively used by other

scholars. These functions generate a nonlinear monotone variational inequality on a certain L^p space, where p is determined by the power law of the cost functions. The cost and demand price functions for the five-firm case in [23] are given by:

$$\begin{aligned} f_i(q_i) &= c_i q_i + \frac{b_i}{b_i + 1} k_i^{-1/b_i} q_i^{\frac{b_i+1}{b_i}}, \quad i = 1, \dots, 5 \\ p(Q) &= 5000^{1/1.1} Q^{-1/1.1}, \quad Q = \sum_{i=1}^5 q_i. \end{aligned}$$

The values of the parameters c_i, k_i, b_i in [23] alongwith our upper bounds for the q_i are given Table 1. An approximate solution of the problem obtained by a projection method is given in [26] as $(q_1, q_2, q_3, q_4, q_5) = (36.937, 41.817, 43.706, 42.659, 39.179)$.

Table 1: Parameter values for the numerical example

i	1	2	3	4	5
c_i	10	8	6	4	2
k_i	5	5	5	5	5
b_i	1.2	1.1	1.0	0.9	0.8
\bar{q}_i	100	100	100	100	100

Before introducing random parameters in the above functions we note that the demand price becomes unbounded when the total quantity Q approaches 0 (commodity is scarce). Although the solution $Q^* = 0$ is never met in most examples, in order to deal with a well behaved function we consider the functional form:

$$p(Q) = 5000^{1/1.1} (Q + e)^{-1/1.1},$$

where e is a small positive parameter which determines the maximum price the consumer can pay when the commodity is very scarce. In our model, we add a random perturbation $r(\omega)$ to c_i , and we modulate the price function by a random function $S(\omega)$.

Thus, for the general case of m firms, we introduce cost functions given by:

$$f_i(\omega, q_i) = [c_i + r(\omega)] q_i + \frac{b_i}{b_i + 1} k_i^{-1/b_i} q_i^{\frac{b_i+1}{b_i}}, \quad (21)$$

where b_i, c_i, k_i are positive parameters, and demand price functions:

$$p(\omega, Q) = [S(\omega)]^a \frac{1}{(Q + e)^a}, \quad (22)$$

where $0 < \underline{s} < S(\omega) < \bar{s}$, and a is a parameter such that $0 < a < 1$ ($a = 1/1.1$ in [23]).

With these functions we can build the Carathéodory function F which defines the variational inequality through:

$$F_i(\omega, q) = c_i + r(\omega) + k_i^{-1/b_i} q_i^{1/b_i} + a[S(\omega)]^a \frac{q_i}{(Q + e)^{a+1}} - \frac{[S(\omega)]^a}{(Q + e)^a}, \quad i = 1 \dots m. \quad (23)$$

We also use the notation $F_i(\omega, q) = G_i(\omega, q) + H_i(\omega, q)$, where G_i represents the sum of the first three terms in (23), while H_i is the rest of the sum, which contains the price function. The monotonicity of F is analyzed in the following:

Theorem 6.1 *The function $F(\omega, \cdot)$ defined by (23) is strictly monotone in \mathbb{R}_+^m , for all $\omega \in \Omega$ and for all fixed values of the parameters therein.*

Proof. Let us observe that the functions $k_i^{-1/b_i} q_i^{1/b_i}$ are strictly increasing for all i , hence the operator $G(\omega, \cdot)$ is strictly monotone on \mathbb{R}_+^n , for all ω .

In order to study the monotonicity properties of H , we preliminary notice that the function $p : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ defined in (22) has strictly positive second derivative (w.r.t. Q):

$$p''(\omega, Q) = \frac{a(a+1)[S(\omega)]^a}{(Q+e)^{a+2}},$$

(recall that $0 < \underline{s} \leq S(\omega)$), therefore $p(\omega, \cdot)$ is strictly convex for all $Q \geq 0$.

Let us now consider the function $Qp(\omega, Q) = [S(\omega)]^a \frac{Q}{(Q+e)^a}$ which is strictly concave on \mathbb{R}_+^n , for each value of ω , with second derivative given by:

$$[Qp(\omega, Q)]'' = [S(\omega)]^a \frac{a(a-1)Q - 2ae}{(Q+e)^{a+2}} < 0$$

($0 < a < 1$). Hence, we get:

$$-2p'(\omega, Q) > Qp''(\omega, Q) \quad (24)$$

which will be exploited in the sequel.

To prove the strict monotonicity of $H(\omega, \cdot)$ for all ω we compute its Jacobian matrix:

$$\begin{aligned} J_{ij}(\omega, Q) &= -p'(\omega, Q) - q_i p''(\omega, Q), \text{ if } i \neq j \\ J_{ii}(\omega, Q) &= -2p'(\omega, Q) - q_i p''(\omega, Q). \end{aligned}$$

It is useful to decompose J as follows:

$$J(\omega, Q) = -p'(\omega, Q)\mathbf{1} - p'(\omega, Q)I - p''(\omega, Q)(q_i)_{ij},$$

where $\mathbf{1}$ denotes the $m \times m$ matrix with each entry equal to 1, I is the identity matrix and the matrix $(q_i)_{ij}$ has each entry of the row i equal to q_i . We prove that $J(\omega, Q)$ is positive definite for all ω and for all $Q \geq 0$ by studying the quadratic form

$$T(\omega, Q)(h) = h^T J(\omega, Q)h, \quad h \in \mathbb{R}^m.$$

From the decomposition of the $J(\omega, Q)$ we then get:

$$\begin{aligned} T(\omega, Q)(h) &= -p'(\omega, Q) \left(\sum_{i,j=1}^m (\mathbf{1})_{ij} h_i h_j + \sum_{i,j=1}^m (I)_{ij} h_i h_j \right) - p''(\omega, Q) \sum_{i,j=1}^m (q_i)_{ij} h_i h_j \\ &= - \left\{ p'(\omega, Q) \left[\left(\sum_{j=1}^m h_j \right)^2 + \|h\|^2 \right] + p''(\omega, Q) \left(\sum_{j=1}^m h_j \right) \left(\sum_{j=1}^m q_j h_j \right) \right\}. \end{aligned}$$

Now, if $h \neq (0, \dots, 0)$, from (24) we get the strict inequality:

$$2T(\omega, Q)(h) > p''(\omega, Q) \left\{ Q \left[\|h\|^2 + \left(\sum_{j=1}^m h_j \right)^2 \right] - 2 \left(\sum_{j=1}^m h_j \right) \left(\sum_{j=1}^m q_j h_j \right) \right\}.$$

Given that $p(\omega, \cdot)$ has strictly positive second derivative, it suffices to prove that the quantity in curly brackets is nonnegative.

Thus, let $h \in \mathbb{R}^m$ with $\sum_{j=1}^m h_j \geq 0$ (the case where $\sum_{j=1}^m h_j \leq 0$ can be analyzed along the same lines), so that:

$$\left(\sum_{j=1}^m h_j \right) \left(\sum_{j=1}^m q_j h_j \right) \leq h_{j_{\max}} \left(\sum_{j=1}^m h_j \right) \left(\sum_{j=1}^m q_j \right) \leq Q h_{j_{\max}} \left(\sum_{j=1}^m h_j \right),$$

where $h_{j_{\max}}$ is and index such that $h_{j_{\max}} \geq h_j, \forall j = 1 \dots m$ and without loss of generality we can assume $j_{\max} = m$ in the sequel. We get then:

$$\begin{aligned} 2T(\omega, Q)(h) &> Q p''(\omega, Q) \left\{ \left[\|h\|^2 + \left(\sum_{j=1}^m h_j \right)^2 \right] - 2 h_m \sum_{j=1}^m h_j \right\} \\ &= Q p''(\omega, Q) \left[\sum_{j=1}^{m-1} h_j^2 + h_m^2 + \left(\sum_{j=1}^{m-1} h_j + h_m \right)^2 - 2 h_m \sum_{j=1}^m h_j \right] \\ &= Q p''(\omega, Q) \left[\sum_{j=1}^{m-1} h_j^2 + \left(\sum_{j=1}^{m-1} h_j \right)^2 \right] \geq 0. \end{aligned}$$

Thus, $T(\omega, Q)(h) > 0, \forall \omega \in \Omega, \forall Q \in \mathbb{R}_+^m, \forall h \neq (0, \dots, 0)$. \square

Now, let us consider the case $m = 5$ with the data as in Table 1. The function F , defines a Nemitsky operator between Lebesgue spaces, as explained in the previous sections. To be precise, since the exponents b_i in the cost functions vary from 0.8 to 1.2, we select $p = 1 + 1/0.8$ so that the Nemitsky operator associated to F maps functions $u \in L^{9/4}$ into $u \in L^{9/5}$. Moreover, we let random parameters $r(\omega)$ and $S(\omega)$ to have truncated normal distributions as follows:

$$\begin{aligned} r &\sim -0.5 \leq N(0, 0.25) \leq 0.5 \\ s &\sim 4950 \leq N(5000, 10) \leq 5050 \end{aligned}$$

while fixing parameter e at 0.0001. Mean values $E(u)$ of $u(r, s) = (u_1, u_2, u_3, u_4, u_5)$ obtained by numerical approximations are presented in Table 2 where n_r and n_s stand for number of discretization points for intervals $[-0.5, 0.5]$ and $[4950, 5050]$ respectively.

7 Conclusions

In this article we considered Nash equilibrium problems in Lebesgue spaces with probability measure and derived their equivalent variational inequality formulation. As a

Table 2: Mean values of $u_i, i=1, \dots, 5$

	$E(u_1)$	$E(u_2)$	$E(u_3)$	$E(u_4)$	$E(u_5)$
$(n_r, n_s) = (200, 20000)$	36.8855	41.7615	43.6448	42.5972	39.121
$(n_r, n_s) = (400, 40000)$	36.913	41.7928	43.6776	42.6294	39.1506

specific application, we proposed a model of oligopolistic market with uncertain data to which the recent theory of random variational inequality ([17, 19]) was applied. We also illustrated our model and the approximation procedure by means of a class of utility functions which yield to nonlinear monotone random variational inequalities.

Further developments of our approach can be done in several directions: other type of probabilistic constraints could be considered instead of the “robust” pointwise constraints (see e.g. [12]); an extension of our numerical method, for example through parallelization, is desirable and would permit the treatment of problems with a larger number of independent random variables; at last, the theory and computation of the stochastic Lagrange multipliers associated to SNEPs in Lebesgue spaces is a topic that has been addressed only recently ([20]) in a simplified model and only from a theoretical point of view.

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